PART : MATHEMATICS

1. Let α, β are roots of quadratic equation and $P_n = \alpha^n + \beta^n$, $P_{10} = 123$, $P_9 = 76$, $P_8 = 47$ and $P_1 = 1$ the quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

(1) $x^2 + x - 1 = 0$
(2) $x^2 - x + 1 = 0$
(3) $x^2 + x + 1 = 0$
(4) $x^2 - x - 1 = 0$

Ans. **(1)**

Sol. $P(x) = ax^2 + bx + c$, $P_1 = 1 \Rightarrow \alpha + \beta = 1 \Rightarrow -\frac{b}{a} = 1 \Rightarrow a = -b$

$aP_{10} + bP_9 + cP_8 = 0$
$123a + 76b + 47c = 0$
$47a + 47c = 0$
$a = -c = -b$
So, equation $x^2 - x - 1 = 0.$
$\alpha + \beta = 1, \alpha\beta = -1$

Sum $= \frac{1}{-1}$ and Prod $= \frac{1}{-1}$
$x^2 - (-1)x + (-1) = x^2 + x - 1 = 0.$

2. The total number of 10 digits numbers formed by only {0, 1, 2} where 1 should be used atleast 5 times and 2 should be used exactly three times, is

Ans. **(2892)**

Sol.

0 digit	1 digit	2 digit
2	5	3
1	6	3
0	7	3

numbers when 0 is used two times
$= \frac{9!}{4!\,3!\,2!} + \frac{9!}{2!\,5!\,2!}$
when 0 is used 1 time.
$\frac{10!}{1!\,6!\,3!} - \frac{9!}{6!\,3!}$
$= \frac{10!}{7!\,3!}$
Total = 2892

3. If $a_1, a_2, a_3, \ldots\ldots$ is an A.P. and $\sum_{k=1}^{12} a_{2k-1} = -\frac{72}{5} a_1$ and $\sum_{k=1}^{n} a_k = 0$ then value of n is

(1) 8 (2) 9 (3) 10 (4) 11

Ans. **(4)**

Sol. $\sum_{k=1}^{12} a_{2k-1} = -\frac{72}{5} a_1$

$\frac{12}{2}[2a_1 + 11(2d)] = -\frac{72}{5} a_1$

$12a_1 + 132d = -\frac{72}{5} a_1$

$60a_1 + 660d = -72a_1$

$132a_1 + 660d = 0$

$2a_1 + 10d = 0$

$a_1 + 5d = 0$

$\sum_{k=1}^{n} a_k = 0$

$\frac{n}{2}[2a_1 + (n-1)d] = 0$

$2a_1 + (n-1)d = 0$

$-10d + (n-1)d = 0$

$-10 + n - 1 = 0$

$n = 11$

4. If $\underline{|50}$ is exactly divisible by 3^n then find maximum value of n.

(1) 20 (2) 22 (3) 18 (4)16

Ans. **(2)**

Sol. $E(3) = \left[\frac{50}{3}\right] + \left[\frac{50}{3^2}\right] + \left[\frac{50}{3^3}\right] = 16 + 5 + 1 = 22$

$\therefore$ Exponent of 3 in $\underline{|50}$ 22.

$\therefore \underline{|50} = 3^{22}.\lambda$ $\therefore$ $n = 22$.

5. Let $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, f(x) has maxima at $x = p$ and minima at $x = q$ such that $p^2 = q$ then find f(3).

(1) 37 (2) 38 (3) 40 (4) 45

Ans. **(1)**

Sol. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$f'(x) = 6x^2 - 18ax + 12a^2$

$f'(x) = 6(x^2 - 3ax + 2a^2) = 6(x-a)(x-2a)$

$\therefore$ $x = p = a$ and $x = q = 2a$

Given $p^2 = q$

$\therefore$ $a^2 = 2a$

$\Rightarrow$ $a = 0$ or 2

$\Rightarrow$ $a = 2$ $(a \neq 0)$

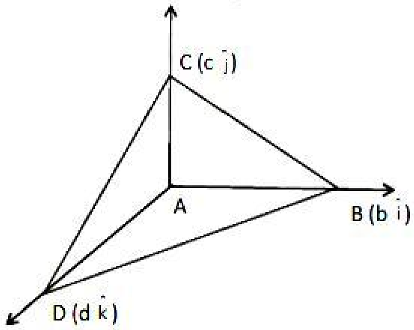
$\therefore$ $f(3) = 37$

6. Let ABCD is a tetrahedron in which $\angle BAC$, $\angle CAD$, $\angle BAD$ is at right angle and area of faces. ABC, BAD & CAD are 5, 7 and 6 respectively then area of face BCD is –

(1) $\sqrt{55}$ (2) $\sqrt{220}$ (3) $\sqrt{110}$ (4) $\sqrt{340}$

Ans. **(3)**

Sol. Let Consider A at origin and B, C, D at x, y & z axes



now

$$\text{ar } \Delta ABC = \left|\frac{1}{2}bc\right| = 5$$

$$\text{ar } \Delta ABD = \left|\frac{1}{2}bd\right| = 7$$

$$\Delta ACD = \left|\frac{1}{2}cd\right| = 6$$

Now area of ΔBCD

$$= \frac{1}{2}\left|\overrightarrow{BC} \times \overrightarrow{BD}\right| =$$

$$\overrightarrow{BC} \times \overrightarrow{BD} = (c\hat{j} - b\hat{i}) \times (d\hat{k} - b\hat{i})$$

$$= \begin{vmatrix} i & j & k \\ -b & c & 0 \\ -b & 0 & d \end{vmatrix}$$

$$= i\,(cd) - j\,(-bd) + k\,(bc)$$

$$\text{are BCD} = \frac{1}{2}\sqrt{(cd)^2 + (bd)^2 + (bc)^2}$$

$$= \frac{1}{2}\sqrt{100 + 196 + 144}$$

$$= \frac{1}{2}\sqrt{440}$$

$$= \sqrt{110}$$

7. Three distinct numbers are selected from set {1, 2, 3, ..., 40}, then find probability that they form an increasing G.P.

(1) $\frac{1}{520}$ (2) $\frac{3}{520}$ (3) $\frac{7}{520}$ (4) $\frac{11}{520}$

Ans. **(1)**

Sol. Given set is {1, 2, 3, ... , 40}

$h(s) = {}^{40}C_3$

a, b, c is G.P. $\therefore b^2 = ac$

a = 1 then c = 4, 9, 16 , 25, 36

a = 2 then c = 8, 18, 32

a = 3 then c = 12, 27

a = 4 then c = 9, 16, 25, 36

a = 5 then c = 20

a = 6 then c = 24

a = 7 then c = 28

a = 8 then c = 32

a = 9 then c = 36

Total cases = 19 = h (A)

$\therefore P(A) = \frac{h(A)}{h(s)}$

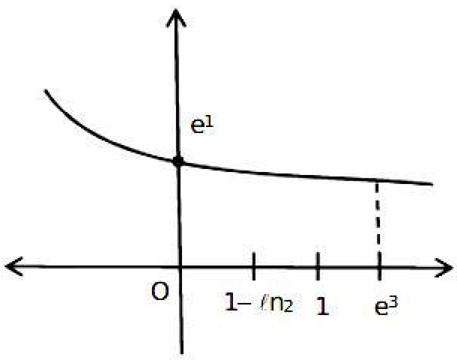
$= \frac{19}{{}^{40}C_3} = \frac{19 \times 3.2.1}{40.39.38}$

$= \frac{1}{520}$

8. $\int_0^{e^3} \left[\frac{1}{e^{x-1}}\right] dx = \alpha - \ell n2$ then α is equal to (where [x] denotes greatest integer less the or equal to x).

(1) 2 (2) 3 (3) 4 (4) 5

Ans. **(1)**



Sol.

$\int_0^{1-\ell n_2} 2dx + \int_{1-\ell n_2}^{1} 1dx + \int_1^{e^3} 0dx$

$= (2x)_0^{1-\ell n_2} + (x)_{1-\ell n_2}^{1} + 0$

$= 2 - 2\ell n2 + 1 - (1 - \ell n2)$

$= 2 - \ell n2 = \alpha - \ell n2$

So, $\alpha = 2$

9. The term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}}+1-x^{\frac{1}{3}}}-\frac{x-1}{x-\sqrt{x}}\right)^{10}$ is

(1) 252 (2) 60 (3) 210 (4) 45

Ans. **(3)**

Sol. $\left(\left(x^{\frac{1}{3}}+1\right)-\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10}$

$\left(x^{\frac{1}{3}}-\frac{1}{x^{\frac{1}{2}}}\right)^{10}$

$T_{r+1} = {}^{10}C_r\, x^{\frac{r}{3}}, \left(-\frac{1}{x^{\frac{1}{2}}}\right)^{10-r}$

$\frac{r}{3}-\left(\frac{10-r}{2}\right)=0$

$2r - 30 + 3r = 0$

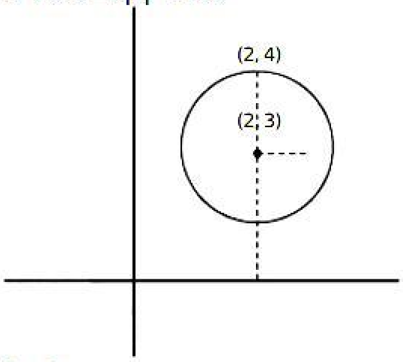
$r = 6$

${}^{10}C_6 = {}^{10}C_4 = 210$

10. If $|z| = 1$ and $\frac{2+k^2z}{\bar{z}+k} = kz$ where $k \in R$ then the maximum value of $k + k^2 i$ from $|z-(2+3i)| = 1$ is

Ans. **(2)**

Sol. $2 + k^2z = k|z|^2 + k^2z$



$k = 2$

$2 + 4i = 2$

11. Area bounded by region $\{(x, y) : |4-x^2| \le y \le x^2 \mid x \ge 0 \mid y \le 4\}$ is

(1) $\frac{1}{3}\left[40\sqrt{2}-48\right]$

(2) $\frac{1}{3}\left[40\sqrt{2}+48\right]$

(3) $\frac{1}{2}\left[40\sqrt{2}-48\right]$

(4) $\frac{1}{2}\left[40\sqrt{2}+48\right]$

Ans. **(1)**

Sol. Solving $4 - x^2 = x^2$

$x = \pm\sqrt{2}$

Required region is as $x \ge 0$

$$= \int_{\sqrt{2}}^{2} x^2-(4-x^2)dx + \int_{2}^{2\sqrt{2}} 4-(x^2-4)dx$$

$$= \int_{\sqrt{2}}^{2} (2x^2-4)dx + \int_{2}^{2\sqrt{2}} (8-x^2)dx$$

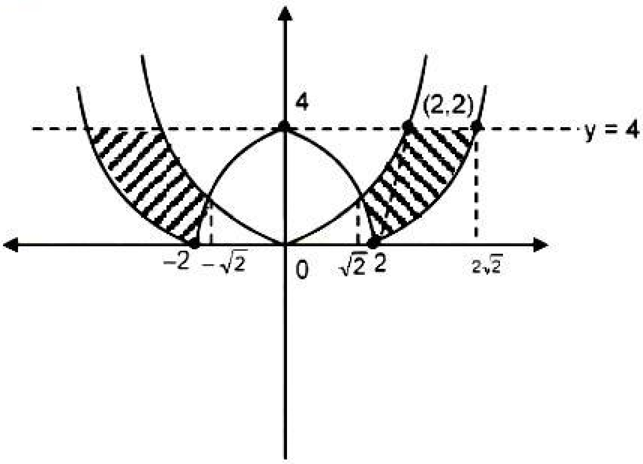
$$= \left[\left(2\frac{x^3}{3}-4x\right)_{\sqrt{2}}^{2}\right] + \left[8x-\frac{x^3}{3}\right]_{2}^{2\sqrt{2}}$$

$$= \left[\left(\frac{16}{3}-8\right)-\left(\frac{4\sqrt{2}}{3}-4\sqrt{2}\right)\right] + \left[\left(16\sqrt{2}-\frac{16\sqrt{2}}{3}\right)-\left(16-\frac{8}{3}\right)\right]$$

$$= \left[\frac{-8}{3}+\frac{8}{3}\sqrt{2}\right] + \left[\frac{32\sqrt{2}}{3}-\sqrt{2}\right]$$

$$= \frac{1}{3}\left[8\sqrt{2}-8+32\sqrt{2}-40\right]$$

$$= \frac{1}{3}\left[40\sqrt{2}-48\right]$$



12. If $\lim\limits_{x\to 0} \dfrac{x^2 \sin(\alpha x) - (\gamma - 1)e^{x^2}}{\sin(2x) - \beta x} = 3$ then the value of $\beta + \gamma - \alpha$ is

(1) –1 (2) 7 (3) 4 (4) 3

Ans. **(2)**

Sol. $$\lim\limits_{x\to 0} \frac{x^2(\alpha x - \frac{\alpha^3 x^3}{31} + ...) - (\gamma - 1)[1 + x^2 + \frac{x^4}{21} + ...]}{2x - \frac{(2x)^3}{31} + ... - \beta x}$$

$$\lim\limits_{x\to 0} \frac{-(\gamma - 1) - (\gamma - 1)x^2 + \alpha x^3 + \frac{(\gamma - 1)}{21}x^4 + ...}{(2 - \beta)x - \frac{8}{6} - x^3 + ...} = 3$$

$\gamma = 1, \beta = 2, \dfrac{\alpha}{-4/3} = 3, \alpha = -4$

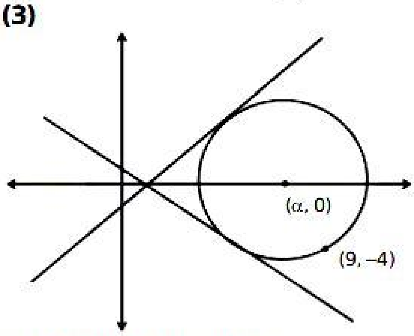
$\beta + \gamma - \alpha = 2 + 1 - (-4) = 7$

13. Two circles touches line $x + y = 3$ & $x - y = 3$ also passes through (9, –4) then absolute difference of radius of two circles is

(1) $\sqrt{90}$ (2) $\sqrt{72}$ (3) $\sqrt{80}$ (4) $\sqrt{120}$

Ans. **(3)**

Sol.



Both circle touches lines

$x + y = 3$ and $x - y = 3$

so centre lies on x-axis. Let centre $(\alpha, 0)$ let radius r

$(x - \alpha)^2 + y^2 = r^2$

it passes through (9, –4)

$(9 - \alpha)^2 + 16 = r^2$ ______(1)

also it touches $x + y = 3$

so $\left|\dfrac{\alpha + 0 - 3}{\sqrt{2}}\right| = r$

$\alpha - 3 = \pm\sqrt{2}\, r$

$\alpha = \pm\sqrt{2}\, r + 3$

put in (1)

$(9 - (\pm\sqrt{2}\, r + 3))^2 = r^2 - 16$

$(36 + 2r^2 \pm 12\sqrt{2}\, r) = r^2 - 16$

$r^2 \pm 12\sqrt{2}\, r + 52 = 0$

Let radius r_1 & r_2

$(r_1 - r_2) = \sqrt{(12\sqrt{2})^2 - 4 \times 52} = \sqrt{288 - 208} = \sqrt{80}$

14. Let P is any point on ellipse $\frac{x^2}{18}+\frac{y^2}{9}=1$ if S_1 and S_2 are focii of ellipse then product of maximum and minimum value of product of length PS_1 & PS_2 is –

Ans. **(162)**

Sol. Let P ($3\sqrt{2}$ cosθ, 3sinθ)

PS_1 & PS_2 = $(a - ex_1)(a - ex_2)$

$= a2 - e^2x_1^2$

$= 18 - \frac{1}{2} \times (18\cos^2\theta)$

$= 18 - 9\cos^2\theta$

So maximum & minimum values are 18 & 9

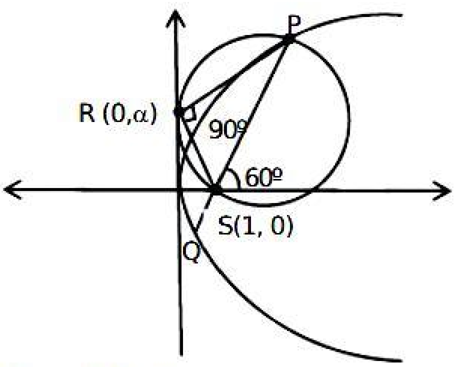
$= 18 \times 9 = 162$

15. Focal Chord PQ of the parabola $y^2 = 4x$ makes an angle 60º with positive x-axis where P in I[st] quadrant. A circle is drawn with SP as diameter touches the tangent at the vectex of the parabola at (0, α) the α^2 is (where S is focus of parabola)

(1) 1 (2) 2 (3*) 3 (4) 4

Ans. **(3)**

Sol.



Let P(t^2, 2t)

$m_{sp} = \frac{2t-0}{t^2-1} = \tan 60º$

⇒ $\sqrt{3}t^2 - 2t - \sqrt{3} = 0$

⇒ $(\sqrt{3}t+1)(t-\sqrt{3}) = 0$

⇒ $t = \sqrt{3}$

∴ Point P is $(3, 2\sqrt{3})$

∴ tangent at P is

$2\sqrt{3}y = 2(x+3)$

at R x = 0

∴ $\alpha = \sqrt{3}$

∴ $\alpha^2 = 3$